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Berry's phase and space symmetries

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Abstract
We study the role played by space symmetries in the context of the Berry phase. It is found that the parameter space may be expanded to include non-compact spaces, that minimal surfaces may be associated with the Berry magnetic field and that, in S^3, the Hopf invariant may be used to classify these surfaces and fields up to a gauge transformation. A number of examples are discussed and some evidence is offered for the existence of Berry phases that are not of the monopole type.

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1. Introduction

Already two decades ago, Berry announced the discovery of a topological phase generated by quantum mechanical systems dependent on slowly varying external parameters as they evolved along cyclic paths about a degeneracy of the quantum levels [1]. During the intervening years, a substantial amount of rethinking of old known ideas (e.g. the Pancharatnam phase, the Born–Oppenheimer approximation, the Aharonov–Bohm effect) as well as related investigations of recent theories (e.g. anomalies, fractional statistics) has given rise to a veritable body of knowledge on the role played by geometry and topology in the understanding of physical phenomena. Almost immediately it was also realized that the Berry phase appears under other guises in classical systems as well, thus extending the context in which it is relevant. In the meantime, experimentalists have kept pace with theoretical developments, verifying many predictions along the way [2, 3].

Two very striking features of the Berry phase are apparent in almost every work on the subject. First is the fact that the relevant physical (parameter) space is compact and multiply connected due to the existence of degeneracies. Second is the natural description of the dynamical process in terms of parallel transport. Both are well-known and well-studied mathematical concepts which have forced physicists to deepen their mathematics background. In this work, we would like to show that symmetries play an important role and that the insight gained from them provides complementary input in understanding the dynamics of the Berry phase.

To clarify the context of our discussion, we point out first that a concrete way in which the Berry phase arises is through the observation of a quantum system, one of whose dynamical variables is coupled to another system which, in turn, undergoes cyclic motion at a much slower rate. The interplay between the 'fast' and 'slow' dynamics leads to the Berry phase if certain conditions are satisfied. This physical picture is very satisfying and lends itself to direct dynamical considerations through ordinary perturbation theory or path-integral analysis. Indeed we will appeal to this paradigm in our discussion below. But this picture may also be compared with the local circulation (vorticity) of fluid elements as they are carried along by a global fluid velocity field. A symmetry which arises here finds its origin in the dragging of the circulation along with the fluid. The analogy is apt, for the circulation, like the magnetic field to which the Berry phase is ascribed, is divergence-free as well. In a related vein, in the physics of perfectly conducting plasmas, this scenario also emerges as the magnetic field is 'frozen' and carried along by the plasma. Such magnetic fields are termed 'force-free' since, as a consequence of the infinite conductivity of the fluid, the electric field vanishes in the frame of the fluid and the fluid moves with zero Lorentz force bearing upon it [4, 5]. In as much as the Berry phase is seen as a phenomenon of geometric origin, the absence of force in these fluids hints at a similar 'geometric' state of affairs.

Let us quickly derive the symmetry condition for the above-mentioned systems from their equations of motion. Consider an inviscid incompressible fluid described by a velocity field $v = v_i(\partial/\partial x_i)$, with uniform density (set at unity). Euler's equation may be written as

$$(\partial/\partial t + \boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \nabla(p+\Omega), \tag{1.1}$$

where Ω is a conservative potential and p is the pressure. On introducing the velocity 1-form $u = u_i\,\mathrm{d}x^i$, we may rewrite the above in differential-forms notation as

$$L_X u = -\mathrm{d}\big(p+\Omega+\tfrac{1}{2}v^2\big), \tag{1.2}$$

where L_X is the Lie derivative with respect to the vector field $X = \partial/\partial t + \boldsymbol{v}$ and d is the exterior derivative. For a perfectly conducting incompressible fluid the electric field vanishes in the frame of the fluid, i.e. $\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} = 0$ and Maxwell's equation $\mathrm{d}\boldsymbol{B}/\mathrm{d}t = -\nabla\times\boldsymbol{B}$ yields, in forms notation, $\mathrm{d}B/\mathrm{d}t = -L_X B$, where $B = \mathrm{d}A$ is the magnetic 2-form, and A the gauge 1-form. For steady-state fields this gives us $L_X A =$ gradient. From this and equation (1.2), we conclude that the most basic fact about fluids is that the Lie derivative of a suitable 1-form is exact:

$$L_X A = \mathrm{d}f, \tag{1.3}$$

X being a divergence-free field and A the relevant 1-form describing the field with f a smooth 0-form. We will see in the next section that, without appealing to dynamical considerations, the Berry phase may be described by a similar equation, and in section 3, that equation (1.3) can serve as the starting point for generalizing the above considerations.

For completeness we review very briefly some background material on the geometry of submanifolds which will be useful in our discussion [6]. For simplicity, take a smooth surface M in R^3. The tangent plane to M at a point P is denoted by $T_P M$ and, again at this point, the normal to $T_P M$ is denoted by $N(P)$. We pass a plane Π, called the normal section, normal to $T_P M$ and hence containing $N(P)$. Let a curve γ on M pass through P with unit tangent vector v at that point. The curvature vector $k(v)$ of γ in the direction v is the acceleration vector under motion along γ with unit speed. It follows that $k(v)$ is collinear with the normal. Since $k(v)$ is a function of the circle formed by all the directions v, it has minimum and maximum lengths k_1 and k_2 for two directions of v. These directions turn out to be orthogonal to each other. We call the average $H = \frac{1}{2}(k_1 + k_2)$ the mean curvature and $H\cdot N(P)$ is the mean curvature vector of M at P.

If α is a curve in M, the velocity is α' and the acceleration $\mathrm{D}_{\alpha'}\alpha'$ where D_V is the natural (Levi-Civita) covariant derivative with respect to the vector field V. Then according to our discussion, the length of the curvature vector is $\langle \mathrm{D}_{\alpha'}\alpha', N(\alpha)\rangle$, where $\langle\cdot,\cdot\rangle$ is the scalar product and α' is understood to have unit length. It is not difficult to see that the acceleration α'' in R^3 of a particle moving on M is the sum of its acceleration $\ddot{\alpha}$ in M and $\langle \mathrm{D}_{\alpha'}\alpha', N\rangle$.

The importance of the mean curvature springs from the following geometrical fact: just as the curvature gives the rate of change of length of an evolving curve, the mean curvature measures the rate of change of area of an evolving surface. In fact, the variation of surface area S with respect to a vector field V is given by $-2\int_S V\cdot HN\,\mathrm{d}\sigma$. Thus, for example, a soap film would tend to move in the direction of positive mean curvature in order to decrease area. When the mean curvature vanishes, we say we have a minimal surface. If one imagines a surface made up of rubber bands stretched out in all directions, then on a minimal surface the forces due to the rubber bands balance out and the surface need not move to reduce tension. Nevertheless, most minimal surfaces are not stable since this requires the second variation of area to vanish as well. In this work, we will show that a direct analogy between minimal surfaces (e.g. soap films in gravity-free space since the mean curvature is proportional to the pressure difference across the bubble surface) and the Berry phase can be established.

The paper is structured as follows: we give a brief review of the Berry phase in section 2 and develop the discussion on the symmetries associated with it. We discuss the general properties of the gauge fields associated with the Berry phase in section 3, deducing a number of them with the aid of the symmetry conditions. In section 4 the equation describing topological Berry fields in R^3 is obtained. A similar effort is made for fields in S^3 in section 5, where we observe that, possibly, an infinite variety of Berry phases might be available in S^3. In section 6, we describe an example of the Berry phase which corresponds to the flat Clifford torus. We show in section 7 that gauge fields having the same Hopf invariant are gauge equivalent. The issue of stability is briefly discussed in section 8. Finally, concluding remarks are given in section 9.

2. Geometric setting of the Berry phase

In this section, we explain the origin of the symmetries exhibited by the Berry phase. In the course of our exposition we also give a brief review of the topological phase in a geometric setting.

Let us start with a Hermitian Hamiltonian $H(\lambda)$ which depends on time through a set of parameters λ describing a Riemannian parameter manifold M. Let us denote the collection of well-behaved fermion (field) states by Ψ. These fields can also be seen as a mapping from M to a state space N which will be assumed to be arcwise connected [7]. If we identify Ψ with a point in N, then the evolution of a fermion state may be represented by a continuous path in this space. In the adiabatic limit, we say that an initial eigenstate $|\Psi(0)\rangle = |n(\lambda(0))\rangle$ corresponding to the nth level of the Hamiltonian continues to remain in this level as λ varies continuously and adiabatically. After a closed loop has been traversed in a time t in M, the fermion state evolves, not to $|n(\lambda(t))\rangle$, as one might naively expect, but to $|\Psi(t)\rangle = |n(\lambda(t))\rangle\,\mathrm{e}^{\mathrm{i}\gamma_n}$, where γ_n is the topological phase [1].

This picture may be translated into geometric language as follows: elements of N are related to each other by a gauge transformation. We say they are gauge equivalent and denote this equivalence by $\sim$. The set of gauge inequivalent points in N forms a new reduced space $R = N/\sim$. It is not difficult to see that a closed cycle in R need not correspond to a closed loop in N. In quantum mechanics, R is the set of rays in the projective representation

[8, 9]. To understand how the Berry phase arises, we apply the adiabatic approximation to the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(t) = H(\lambda(t))\Psi(t), \tag{2.1}$$

confining ourselves to the nth level defined by

$$H(\lambda)|n(\lambda)\rangle = E_n(\lambda)|n(\lambda)\rangle. \tag{2.2}$$

On introducing the ansatz

$$|\Psi(t)\rangle = \exp\left\{-\mathrm{i}\int_0^t \mathrm{d}t' E_n(\lambda(t'))\right\} \mathrm{e}^{\mathrm{i}\gamma_n}|n(\lambda)\rangle, \tag{2.3}$$

we find, for a closed loop C in M,

$$\gamma_n = \mathrm{i}\oint_C A, \qquad A = \langle\Psi|\mathrm{d}\Psi\rangle, \tag{2.4}$$

where A is the gauge field [1]. It can be shown that γ depends only on the path and not on the parametrization employed [10, 11].

The interpretation of these results becomes transparent if we redefine the Hamiltonian by subtracting the corresponding eigenvalue $E_n(\lambda)$. Then using tildes to distinguish quantities corresponding to this shift, we have

$$\mathrm{Im}\langle\tilde{\Psi}|\mathrm{d}\tilde{\Psi}\rangle = 0, \tag{2.5}$$

so that $A = \langle\tilde{\Psi}|\mathrm{d}\tilde{\Psi}\rangle$ is the anti-Hermitian connection of the nth-level Hilbert subbundle which vanishes along the path actually followed during the adiabatic motion of the system. Vectors satisfying equation (2.5) are said to be horizontal and the connection A is a rule for defining a horizontal subspace anywhere in N. Analogous to the remarks above, a closed curve in R (the base space) need not correspond to a closed path in N. From a mathematical standpoint this path is the lift of the closed loop, and tangent vectors along this path satisfy equation (2.5). We associate the integral $\oint A$ with the lifted path by closing the latter with a vertical segment connecting the endpoints. Although equation (2.5) is satisfied along the lift, it is not true along the vertical segment. Hence $\gamma = \mathrm{i}\oint A$ is nonzero. This then is the origin of the topological phase.

In fact γ is a gauge-invariant quantity. Related curves in N are linked by a gauge transformation: $|\Psi_a\rangle = |\Psi_b\rangle U_{ba}$, where U is unitary. The corresponding transformed connection is given by $A^U = U^\dagger A U + U^\dagger \mathrm{d}U$ and the magnetic field (curvature) $F = \mathrm{d}A + A \wedge A$ turns out to transform covariantly, $F^U = U^\dagger F U$. Thus $\oint A = \int_{\partial^{-1}C} F$ is gauge invariant. After a complete cycle is traversed, states are multiplied by a unitary matrix $\exp(-\oint A)$ which mixes states among themselves in a gauge invariant way. In the usual formulation of the Berry phase, one considers the set N' of wavefunctions of unit norm and discovers that N' is a principal fibre bundle over the projective space of rays. The Berry phase arises from the anholonomy angle associated with parallel transport.

From the manner by which the topological phase is generated we gather that certain symmetries arise naturally. To begin with, let us first observe that a discussion of symmetries requires a rule for comparing fields at different points of M. The simplest one which requires no additional structure on M is that of Lie dragging the field A at a point $\lambda + \Delta\lambda$ to another point λ along a smooth curve. One can then compare $A(\lambda)$ with $A(\lambda + \Delta\lambda)$ in an unambiguous way.

The symmetries we are considering arise as follows: we associate the set of manifold-filling non-intersecting curves along which fields are to be dragged with a vector field X. A given field A is said to possess the symmetry generated by X if the field $A(\lambda + \Delta\lambda)$, when dragged to λ, coincides with $A(\lambda)$.

To convert the above rule into a calculational procedure, we note that the vector field X may also be viewed as the generator of a one-parameter family of diffeomorphisms of M to itself, $x \to x' = x + \varepsilon \cdot X$. In fact, a set of symmetries is a group of motions of the manifold M with which we associate transformations of the field. For a scalar field this transformation is just a translation. Under the mapping then, A is transformed to A' which we define by

$$A'(x) = A(x'). \tag{2.6}$$

From the transformation rule for vectors

$$A'_\mu(x) = \frac{\partial x'^\nu}{\partial x^\mu} A'_\mu(x) = \left(\delta^\nu_\mu + \varepsilon \partial_\mu X^\nu\right)(A_\nu(x) + \varepsilon X^\rho \partial_\rho A_\nu), \tag{2.7}$$

we now define the Lie derivative of A_μ with respect to X by

$$\begin{aligned} L_X A_\mu &= \lim \frac{1}{\varepsilon}(A'_\mu(x) - A_\mu(x)) \\ &= (\partial_\mu X^\rho) A_\rho + X^\rho \partial_\rho A_\mu. \end{aligned} \tag{2.8}$$

Hence a vector field A_μ possesses the symmetry generated by X^μ if $L_X A_\mu$ vanishes. The discussion for more general tensor fields follows a similar development. Each diffeomorphism δ gives rise to a bundle map for every tensor bundle over M. To each local cross section T of the tensor bundle there corresponds a cross section $T(t)$ for each diffeomorphism $\delta(t)$, t being the parameter characterizing the family of diffeomorphisms. The Lie derivative is defined by

$$L_X T = \lim_{t \to 0} \frac{1}{t}(T(t) - T). \tag{2.9}$$

If the Lie derivative vanishes, then we will say that T possesses the symmetry generated by X.

In the case of gauge fields, the requirement $L_X A_\mu = 0$ is overly restrictive on account of the fact that $A^U = U^\dagger A U + U^\dagger \mathrm{d}U$ is gauge equivalent to A. It suffices to weaken our definition above by replacing $A'_\mu(x)$ with $A^U_\mu(x)$ for some particular U. If we set $U = \mathrm{e}^{\mathrm{i}\varepsilon\phi}$ we obtain

$$A^U_\mu(x) = A_u + \varepsilon(\partial_\mu \phi + [\phi, A_\mu]), \tag{2.10}$$

so the above symmetry condition becomes

$$L_X A_\mu = \mathrm{D}_\mu \phi, \tag{2.11}$$

where D_μ is the standard covariant derivative. In as much as we will only concern ourselves with the Abelian case, we replace this by

$$L_X A = \mathrm{d}\phi. \tag{2.12}$$

This then is the general mechanism by which the symmetries of section 1 arise. In the next section, we study the consequences of this equation in connection with the Berry phase. Unlike equation (1.3), condition (2.12) was obtained without recourse to dynamical equations. The above concept of symmetry is discussed within a fibre-bundle context by Harnad, Shnider and Vinet [12].

As an example of the discussion above, we briefly discuss Berry's now classic model [1]. A spin-$\frac{1}{2}$ particle in a magnetic field is governed by the Hamiltonian $H(\boldsymbol{R}) = \boldsymbol{R}\cdot\boldsymbol{\sigma}(R =$ constant). It has eigenvalues $\pm R$. The normalized $+R$ eigenstate may be written as

$$u_+ = [2R(R-R_3)]^{-1/2}\begin{pmatrix} R_1 - \mathrm{i}R_2 \\ R - R_3 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \tag{2.13}$$

which satisfies $(H(\boldsymbol{R}) - R)u_+ = 0$. From equation (2.13) we calculate the gauge potential

$$A = \langle u_+|\mathrm{d}|u_+\rangle = \frac{\mathrm{i}}{2}\frac{R_2\,\mathrm{d}R_1 - R_1\,\mathrm{d}R_2}{R(R-R_3)}, \tag{2.14}$$

with div $A = 0$, which displays the well-known Dirac string singularity. The corresponding 'magnetic field' is

$$B = \mathrm{d}A = \frac{\mathrm{i}}{2}\frac{R_1\,\mathrm{d}R_2\wedge\mathrm{d}R_3 + R_3\,\mathrm{d}R_1\wedge\mathrm{d}R_2 + R_2\,\mathrm{d}R_3\wedge\mathrm{d}R_1}{R^3}. \tag{2.15}$$

This is the standard magnetic monopole field. It is clear that the surface of constant magnetic field is a patch of a sphere S^2. The geometric phase is

$$\gamma = \oint_C \mathrm{d}A = \frac{1}{2}\Omega, \tag{2.16}$$

where Ω is the solid angle subtended by the closed arc C as seen from the point of degeneracy $\boldsymbol{R} = 0$. We can verify directly that equation (2.12) is satisfied by $X = R_1\partial/\partial R_2 - R_2\partial/\partial R_1$, which represents motions along the curve C. The calculation is simplified by observing that A is a connection of the Hopf bundle of S^3 over S^2. In a given gauge, $A = \frac{1}{2}(1+\cos\phi)\,\mathrm{d}\phi$ and $X = \partial/\partial\phi$. Because of the degeneracy at the origin, the parameter manifold has the topology of a compact 2-sphere.

3. General properties

Continuing the development of section 2, we explain the general properties of the gauge field. Initially, we will take a more general point of view by considering a divergence-free $(n-1)$-form α living in a $(2n-1)$-dimensional Riemannian manifold M of fixed orientation. Wherever necessary a metric will be assumed to have been introduced. The case of three dimensions corresponds to $n = 2$. In the standard Berry example one has the Hopf fibration $S^3 \to CP^1$ and the pullback of the Fubini–Study metric may be used. For higher dimensional spaces we refer to Nowakowski and Trautmann [13]. From equation (1.3) and the end of section 2, we therefore start with the equations for an $(n-1)$-form α

$$\mathrm{d}\,{}^*\alpha = 0 \tag{3.1a}$$

$$L_X\alpha = \mathrm{d}h, \tag{3.1b}$$

where h is an $(n-2)$-form, ${}^*\alpha$ is the Hodge dual for α and X is a divergence-free vector field. We employ the conventions of Eguchi, Gilkey and Hanson [14]. We tacitly assume that the quantities involved satisfy suitable continuity and differentiability requirements. From here on, fields obeying equation (3.1) will be called gauge fields. Let us now study the properties of these fields. Our ultimate goal, to be reached in sections 6 and 7, is to show that these gauge fields can be uniquely classified in terms of their Hopf invariants.

First of all, the vector fields X_i form a Lie algebra with respect to the Lie brackets [,] because

$$L_{[X_i,X_j]}\alpha = \big(L_{X_i}L_{X_j} - L_{X_j}L_{X_i}\big)\alpha = \mathrm{d}\big(L_{X_i} - L_{X_j}\big)h, \tag{3.2}$$

and $[X_i, X_j]$ is divergence-free if X_i and X_j are. The vectors X_i constitute an involutive vector space in this sense. Thus several symmetries may be simultaneously considered.

We introduce next the energy E_B and the Hopf invariant S_{AB}

$$\begin{aligned} E_B &= \int_M {}^*B \wedge B \equiv (B, B) \\ S_{AB} &= \int_M \alpha \, \mathrm{d}\alpha \end{aligned} \tag{3.3}$$

where $B \equiv \mathrm{d}\alpha$ is an n-form. S_{AB} is a topological invariant which measures the helicity of fields (see below). Under transformations generated by X, S_{AB} is an invariant, for by equation (3.1)

$$L_X S_{AB} = 0. \tag{3.4}$$

Furthermore, if X is a Killing vector field, then it is known that $L_X^* = {}^*L_X$ so we have

$$L_X E_B = 2 \int {}^*B \wedge L_X B = 0, \tag{3.5}$$

by equation (3.1*b*), i.e., E_B is invariant under flows arising from X (isometries). Since Killing vector fields are divergence-free, we conclude that S_{AB} and E_B are invariant under isometries.

Equation (3.1*b*) translates to $L_X B = L_X \, \mathrm{d}\alpha = 0$. In as much as B is an n-form, it describes an n-surface S defined here as the surface for which B is proportional to the surface area n-form of S, i.e., S is homologically trivial. Most of the interesting submanifolds will be embedded either in R^3 or S^3 and we will assume that this stipulation suffices for the cases of interest. Such surfaces may be termed magnetic surfaces and have been discussed in the literature [15].

We show below that this implies that S corresponds to a minimal surface. A result like this is to be expected since, according to Samuel and Bhandari [16], the Berry phase is the line integral of the connection A evaluated along a geodesic curve. It is natural then to search for manifolds wherein such geodesic curves play a special role. Because of the paucity of submanifolds whose geodesics are also geodesics of the manifold in which they are immersed (i.e., totally geodesic submanifolds), the next logical candidate is a minimal surface. Let us first show that X need not be normal to S. Let $E_1, E_2, \ldots, E_n$ be a local orthonormal frame in S. From a well-known formula [17]

$$(L_X B)(E_1, E_2, \ldots, E_n) = X B(E_1, E_2, \ldots, E_n) - \sum_{i=1}^{n} B(E_1, \ldots, [X, E_i], \ldots, E_n). \tag{3.6}$$

By virtue of our discussion above, the left-hand side of equation (3.6) vanishes. Since $B(E_1, E_2, \ldots, E_n)$ is the volume form and X is volume preserving, the first term on the right-hand side vanishes. It follows that the second term vanishes as well. On the other hand, we may write for the second term on the right-hand side

$$-\sum_i \sum_j B(E_1, \ldots, g([X, E_i], E_j) E_j, \ldots, E_n) = -B(E_1, \ldots, E_n) \sum_i g([X, E_i], E_i) = 0 \tag{3.7}$$

($g(,)$ denotes the metric). If the magnetic field $B(E_1, E_2, \ldots, E_n)$ does not vanish, then X is normal to S and it belongs to the normal subspace of S. However, it may happen that the field vanishes and in this case X need not commute with E_i. We will examine this possibility in section 5.

Now the mean curvature along a normal N to S, denoted by $k(N)$, is obtained by a natural generalization of the discussion of section 1:

$$\begin{aligned} k(N) &= \frac{1}{n}\sum_i \left(\mathrm{D}_{E_i}E_i, N\right) = -\frac{1}{n}\sum_i \left(\mathrm{D}_{E_i}N, E_i\right) \\ &= -\frac{1}{n}\sum_i \left(\mathrm{D}_N E_i + [E_i, N], E_i\right) \\ &= -\frac{1}{n}\sum_i \frac{1}{2}\mathrm{D}_N(E_i, E_i) + \frac{1}{n}\sum_i g([N, E_i], E_i) \\ &= \frac{1}{n}\sum_i g([N, E_i], E_i). \end{aligned} \tag{3.8}$$

In the second line above we had used the fact that $\mathrm{D}_X Y - \mathrm{D}_Y X = [X, Y]$ [6]. Since $[N, E_i]$ vanishes, $k(N) = 0$ and so S is a minimal surface.

Let us restrict ourselves for a moment to three-dimensional space. Then α is a 1-form and we can always choose a gauge such that equation (3.1) becomes $L_X\alpha' = 0$, where α' is the gauge transformed connection 1-form. Making use of the Cartan formula [17] $L_X = i(X)\,\mathrm{d} + \mathrm{d}i(X)$ for the Lie derivative and the fact that X is normal to S, we find that $i(X)\alpha' =$ constant. That is, the projection of the connection on a Killing field is constant. In other words, in 3-space, the rate at which Berry's phase changes is constant along a Killing field. Although a similar statement can be given for general manifolds the interpretation in terms of the Berry phase is not as transparent.

Physically $L_X B = 0$ means the invariance of B under flows generated by X. As the above calculation shows, the Berry magnetic field issues normally through a minimal surface. The magnetic surfaces are analogues of the equipotential surfaces in electrodynamics or fluid mechanics, where these are important in their own right.

Because of our interpretation of B as the volume form of S, the condition that S is a minimal surface is equivalent, by equation (3.3), to the condition that energy is minimum. This is analogous to surface energy in soap bubbles. We can go further by showing that energy is a minimum when

$$\mathrm{d}\alpha = \lambda^*\alpha, \tag{3.9}$$

where λ is a constant. (If we accept equation (3.1a), then λ can only be a constant and in this case it is clear that $E_B = \lambda S_{AB}$.)

To obtain equation (3.9), consider the quantity $E_B - \lambda S_{AB} = \int \alpha\,\mathrm{d}\{(-)^{n*}\mathrm{d}\alpha - \lambda\alpha\}$. On varying with respect to α, the above result readily emerges. To verify that we have minimum field energy, define $I \equiv (\alpha, \alpha)$. S_{AB} satisfies the Schwarz inequality $S_{\mathrm{AB}}^2 \leqslant I \cdot E_B$ with equality holding precisely when equation (3.9) is true. Since S_{AB} is a topological invariant (see below), it follows that equation (3.9) is indeed the condition for minimum energy. (One may add to the right-hand side of equation (3.9) an n-form $^*\mathrm{d}\beta$ but this contributes a total divergence to the field energy and may be ignored.) In the minimal-energy case, one can show directly from equation (3.9) that

$$\Delta\alpha = \lambda^2\alpha, \tag{3.10}$$

where $\Delta = \delta\,\mathrm{d} + \mathrm{d}\delta$ is the Laplacian, and $\delta = (-)^{pn+n+1\,*}\mathrm{d}^*$ is the co-derivative acting on a p-form in n-space.

To relate the present discussion to the Berry phase, let us once more look into the considerations introduced in section 1. Suppose a vector field $\boldsymbol{F}$ is co-moving with a fluid.

The flux of $\boldsymbol{F}$ through a surface element $\mathrm{d}\boldsymbol{\sigma}$ is defined by $f = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{\sigma}$. We say that the field is frozen into the fluid if

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0,$$

where $\mathrm{d}/\mathrm{d}t$ is the convective derivative. We can rewrite this as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial}{\partial t} f + L_u f = 0. \tag{3.11}$$

In the case of a static incompressible flow, this is the analogue of equation (3.1). Hence our defining equation (3.1) implies that the Berry field is a static field frozen into the moving fluid, i.e., the Berry field is frozen into the coordinate frame travelling with the system as it is adiabatically transported about a closed loop in a force-free manner. Let S denote the corresponding magnetic surface, and let us assume further that during adiabatic transport, the boundary of S remains fixed or empty. Then the change in its area as a result of this transformation is $-2\int_S V \cdot HN\, \mathrm{d}\sigma$, where H is the mean curvature (which vanishes for any minimal surface). It follows that the condition that the Berry magnetic field corresponds to the volume form of the minimal surface S ensures that transport introduces no net changes in the area, and therefore no additional expenditure of energy. Thus, during adiabatic transport, energy is both conserved and kept at its minimum value. The helicity is conserved as well.

Finally, we give some remarks about the Hopf invariant. The Hopf invariant gives the linking number of two disjoint manifolds and is a genuine topological invariant [18–22]. In the minimal energy case, if we normalize α such that $(\alpha, \alpha) = 1$, then λ is an integer, namely the linking number. The Hopf invariant classifies manifolds into equivalence classes. We can further appreciate the relevance of S_{AB} as follows. In R^3 let C be a closed curve on a surface S and let the vector field A be everywhere normal to S. Then the loop integral $\int_C A\, \mathrm{d}l$ vanishes. By Stokes' theorem this integral is also equal to $\int_{S'} \nabla \times A \cdot \mathrm{d}\sigma$, where C bounds the surface $S' \subset S$. But this surface integral does not vanish in general unless $A \cdot \nabla \times A = 0$ everywhere. This translates, in simply connected 3-space, to the impossibility of constructing (even locally) surfaces that are normal to A unless the condition $A \cdot \nabla \times A = 0$ holds. We will show in section 7 that this is equivalent to the vanishing of the Hopf invariant. Thus, the non-vanishing of the Hopf invariant signals a nontrivial geometric structure. In the minimal-energy case, equation (3.9) shows that a nontrivial manifold structure is involved. Berry's example at the end of section 2 corresponds to the principal bundle $\pi\colon S^3 \to S^2$ which has nontrivial topology (see section 5 also).

4. Fields in Euclidean space

In this section, we show in a concrete instance how the symmetry requirement (3.1*b*) may be exploited to obtain the equation for topological fields in Euclidean three-dimensional space. It turns out that there is a unique magnetic surface corresponding to a helicoid which emerges, and that there are no such surfaces of finite range in Euclidean space.

To construct an example, consider a spinning charged particle which is carried along a helix C of pitch angle $\alpha = 45^\circ$ and unit radius. The charge is transported in such a way that its spin direction is always tangent to the helical path. Here we will not be concerned with the spin beyond the fact that it fixes the direction of the particle along its path. We can now form an orthonormal triad of vectors $T, U, T \times U$, where T is the unit tangent vector and U is a unit vector orthogonal to T. As the charge winds along C, the triad rotates about T. The angle of rotation is given by

$$\varphi = \int_C U \times T \cdot \frac{\partial}{\partial x_i} U\, \mathrm{d}x_i. \tag{4.1}$$

By analogy with equation (2.4) we may regard $A_i = U \times T \cdot \nabla_i U$ as our gauge potential. For the example at hand, we have $T = (-y, x, 1)/\sqrt{2}$, $U = (x, y, 0)$, $x^2 + y^2 = 1$. Then $A = (y, -x, 1)/\sqrt{2}$ and $\varphi = 2\pi/\sqrt{2}$, per turn. In the general case, $\varphi = 2\pi \sin\alpha$ (α = pitch angle) and the phase anholonomy is $2\pi(1 - \sin\alpha)$. To see the form of the gauge potential more transparently, we perform a gauge transformation: $A' = A + \nabla\Psi$, where $\Psi = -z/\sqrt{2}$ and obtain $A' = (\cos z, -\sin z, 0)/\sqrt{2}$ which is helicoidal. One verifies that equation (2.11) holds once again, this time for $X = y\partial/\partial x - x\partial/\partial y + \partial/\partial z$. Both A and the magnetic field are constant over this helicoid.

Our parameter manifold for this example has the topology of the straight line. This is manifested by the fact that only A_z contributes to φ. We can think of the particle trajectory as a thin ribbon following the helix. As it moves, the particle keeps track of its rotation from its initial position. In this sense, φ is a measure of the total twist of the ribbon. Unlike Berry's example the parameter manifold in the present example is non-compact.

Let us now see how we can deal directly with equation (3.1). Suppose we integrate both sides of equation (3.1*b*) over any patch of the surface S described by $B = \mathrm{d}\alpha$. Then at each point of S, B is a constant times the surface n-form of S. Clearly $L_X \int_{\partial S} \alpha = 0$, where ∂S is the boundary of the patch. Since ∂S is arbitrary, we must have $\int_{\partial S} \alpha = 0$. But in R^3, α is just $f\,\mathrm{d}s$, where f is a function on ∂S constructed from vectors available on S and $\mathrm{d}s$ is the element of arc length on ∂S. Now the only vectors available are the conormal $\hat{n}_\perp$ and an arbitrary fixed vector $\hat{b}$ in R^3. Then we have

$$\int_{\partial S} \hat{n}_\perp \cdot \hat{b}\,\mathrm{d}s = 0. \tag{4.2}$$

Evidently, the form of α is dependent on the particular geometry of S.

Suppose we assume that S is a ruled surface, i.e. a surface of the form $x(t, v) = \alpha(t) + v w(t)$, generated by the family $\{\alpha(t), w(t)\}$, where $\alpha(t)$ is a curve in R^3 and $w(t)$ a non-vanishing vector assigned to $\alpha(t)$ and $(t, v \in R)$. Then, by a theorem in geometry [23], S depends on two parameters u and v such that

$$\hat{n}_\perp = \sigma'(u) + v\delta'(u), \tag{4.3}$$

where σ and δ are vectors satisfying $\delta - \delta = 1$ and $\sigma' - \delta' = 0$ and $'$ denotes differentiation with respect to u. If we choose $b = (0, 0, 1) \equiv \hat{k}$ and slice S with planes at $z = z_1$ and $z = z_2$ parallel to the xy plane, equation (4.2) yields

$$\hat{k} \cdot (\sigma' + v\delta')|_{z_1} = \hat{k} \cdot (\sigma' + v\delta')|_{z_2}, \tag{4.4}$$

which is obviously satisfied if σ' = constant and in the z-direction. Hence we may set $\sigma = (0, 0, u)$ and we find $\delta = (\cos u, \sin u, 0)$. In forms notation, we have

$$\alpha = \cos z\,\mathrm{d}x - \sin z\,\mathrm{d}y, \tag{4.5}$$

with $\mathrm{d}^*A = 0$, which was the example considered above. One can now check that equation (3.1*b*) holds for $X = -x\partial/\partial y + y\partial/\partial x + \partial/\partial z$ and α has linking number $\lambda = 1$ per turn about the z-axis. The surface is a helicoid, the earliest nontrivial minimal surface discovered. In the text of force-free fields this had been treated before by Ferraro and Plumpton [4]. Finally, let us show that the helicoid is the only gauge field in R^3.

First, let us observe that in Euclidean space, topological fields do not span a *compact* manifold. Suppose that S is closed and compact in R^3. Then ∂S is a closed curve in S dividing it into two parts. Since $\hat{b} \cdot \hat{n}_\perp$ cannot vanish on ∂S (i.e., α is nondegenerate), clearly equation (4.2) cannot be satisfied. In the case of a non-compact surface, ∂S may be made up of two curves dividing S into *three* parts. In this instance equation (4.2) may be satisfied as the example above shows.

Now, we may think of the spin as a vector in the tangent space at every point of S. It follows that the Berry phase is a rule by which the tangent space of S may be constructed along a loop. By equation (3.1), we understand that S enjoys a symmetry property by which transport in the direction of the vector field X leaves its volume unchanged. But in R^3 we have a natural vector field, namely the vector field defined by the spin. Because of the affine nature of R^3 it now follows that S must be a ruled surface. But a well-known result in geometry tells us that the only ruled minimal surface in R^3 is a helicoid (Catalan's theorem [24]). Thus there is only one topological field in R^3 and this is the helicoidal field discussed above.

To investigate gauge fields of finite range it is natural to consider those embedded in a sphere. The next two sections discuss this issue.

5. Topological fields on the sphere

In this section we obtain the equation obeyed by gauge fields (described by $(n-1)$-forms) living on the unit sphere $S^3 \subset R^4$. Evidently Berry's example for a spin-$\frac{1}{2}$ system falls into this category. In fact, as shown by Segert [25], the requirement of rotational symmetry of the field on S^3 suffices to determine the field uniquely. This explains why the monopole form of Berry's example is so ubiquitous. However, we will argue that in S^3 an infinite variety of topological fields can exist and in section 6 we give an example that is not of the monopole type.

First, we examine equation (3.1) in a general 3-manifold M^3: A is a 1-form and h a 0-form. Suppose that the magnetic field is non-vanishing so X is normal to S. Since $L_X\, \mathrm{d}A$ vanishes we have $\mathrm{d}i_X\, \mathrm{d}A = 0$ or $i_X\, \mathrm{d}A = \mathrm{d}h'$, where h' is a new 0-form. The condition $h' =$ constant describes level surfaces in M^3 and simultaneously the magnetic surfaces S which are transverse to X. Thus h'^{-1}(constant) are the leaves of a co-dimension one foliation of M^3 corresponding to magnetic 2-surfaces that are everywhere transverse to the vector field X. We assume closed magnetic field lines so that S must be closed and compact. In virtue of the above correspondence then, $h'^{-1}(c)$ are closed and compact leaves.

Next we look into the construction of tangent vectors on magnetic surfaces. At any point of S we can set up a distinguished set of coordinates (x, y, z) and write $X = f(\partial/\partial z)$ where f is a 0-form. Since $\operatorname{div} X = 0$, we have $f = f(x, y)$. If S has non-vanishing Gaussian curvature then the first derivatives of f do not vanish. We may thus form two vector fields $U = f_x(\partial/\partial x)$ and $V = f_y(\partial/\partial y)$ everywhere tangent to S and commuting with each other, $[U, V] = 0$ (notation: $f_x = \partial f/\partial x$). Then the action of the group R^2 on the closed 2-surface S is a torus. This gives us a topological classification of the flow lines on magnetic 2-surfaces. An exception to this is the case of compact and connected 3-manifolds. By Lima's theorem, which states that two commuting vectors in such manifolds must be linearly dependent at some point [26], we conclude that there is only one such vector field in this case.

We also inquire about the possibility of the magnetic field vanishing everywhere. Consider equation (3.7). If the magnetic field vanishes, then $\sum g([X, E_i], E_i)$ need not vanish. Thus X is not normal to the magnetic surface. Now we know that a compact simply connected 3-manifold has rank 1, that is, two commuting vectors must necessarily be linearly dependent somewhere (Lima's theorem). A sufficient condition that a 3-manifold is simply connected is that $A\, \mathrm{d}A$ vanishes. Thus a three-dimensional compact space in which $A\, \mathrm{d}A = 0$ has zero magnetic field everywhere. It is tempting to replace this last condition by that of zero linking number; however, we will be able to show this only in section 8. An example of a system with zero field but nonzero connection is given in section 6.

We now obtain an equation satisfied by gauge fields in S^3. Let X be a Killing field and let $f = \frac{1}{2}\langle X, X\rangle$ be half of its length squared. Guided by the work of section 3 we compute the

Laplacian of f, defined here alternatively as a contraction:

$$\Delta f = \sum_i \langle \mathrm{D}_{E_i} \operatorname{grad} f, E_i \rangle. \tag{5.1}$$

From the definition $\langle \operatorname{grad} f, X\rangle = Xf$, valid for any vector field X, we obtain grad $f = -\mathrm{D}_X X$. Now we invoke the definition of the Riemannian curvature tensor R for a set of vectors V, W and X [6]:

$$\langle \mathrm{D}_V\, \mathrm{D}_X X, W\rangle = \langle \mathrm{D}_{[V,X]} X - R_{VX} X + \mathrm{D}_X\, \mathrm{D}_V X, W\rangle. \tag{5.2}$$

Because X is a Killing vector, i.e. it satisfies $\langle \mathrm{D}_V X, X\rangle + \langle \mathrm{D}_X X, V\rangle = 0$, the last two equations yield

$$\Delta f = -\mathrm{Ric}(X, X) + \mathrm{tr}(\mathrm{D}X, DX), \tag{5.3}$$

where $\mathrm{Ric}(X, X) = \sum_{E_i} \langle R_{XE_i} X, E_i\rangle$ is the Ricci tensor. Equation (5.3) holds for any Killing vector X. On the sphere, however, $X = x^i \partial_j - x^j \partial_i$ so that f has the simple form $\frac{1}{2}(x^i)^2 + \frac{1}{2}(x^j)^2$. Also D$X$ is zero or normal to the sphere so tr(DX, DX) vanishes. Because the sphere S^3 has constant sectional curvature, we have

$$\mathrm{Ric}(X, X) = 2\langle X, X\rangle / r^2, \tag{5.4}$$

where $r = 1$ is the radius of S^3. With f given above, the left-hand side of equation (5.3) may be cast as $\langle \mathrm{D}x^i, \mathrm{D}x^i\rangle + \langle x^i, \Delta x^i\rangle$ and finally we obtain

$$\Delta x^i = -2x^i, \tag{5.5}$$

as the equation for the components occurring in the Killing vector X which generates isometry transformations on the fields. Alternatively, x^i describe the coordinates of the fields embedded in S^3 upon which X operates. (In Euclidean space we have $r \to \infty$, so equation (5.5) is replaced by $\Delta x^i = 0$. One can verify that the helicoid of the previous section satisfies it.)

It is well known that equation (5.5) is the equation of a minimal $n = 2$ surface embedded in S^3 [27]. Thus, we have shown that the gauge fields obeying equation (3.1) and embedded in S^3 correspond to minimal surfaces.

To illustrate this, let us consider Berry's example once more. It is really a Dirac monopole in parameter space. The Dirac vector potential is just the connection of S^3, the principal $U(1)$ bundle over S^2 [28]. If we think of S^3 as a subset of $C^2 = C \times C$, the Killing field is generated by the rotation of one of the complex factors and is tangent to two circles.

We employ real coordinates y^1, y^2, y^3, y^4 on C^2 with $(y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 = 1$ for S^3. The corresponding Dirac (connection) 1-form is

$$A = \frac{1}{\pi}(y^1\, \mathrm{d}y^2 + y^3\, \mathrm{d}y^4). \tag{5.6}$$

In S^3, the volume form is proportional to

$$\sigma = y^1\, \mathrm{d}y^2 \wedge \mathrm{d}y^3 \wedge \mathrm{d}y^4 - y^2\, \mathrm{d}y^1 \wedge \mathrm{d}y^3 \wedge \mathrm{d}y^4 + y^3\, \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^4 - y^4\, \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}y^3,$$

so we find $\mathrm{d}A = {}^*A$, and of course $\mathrm{d}^*A = 0$. Moreover,

$$\Delta A = A \tag{5.7}$$

and the Hopf invariant is

$$\begin{aligned} \int_{S^3} A\, \mathrm{d}A &= \frac{2}{\pi^2} \int_{S^3} y^1\, \mathrm{d}y^2\, \mathrm{d}y^3\, \mathrm{d}y^4 \\ &= \int_0^\pi \int_0^\pi \int_0^{2\pi} \sin^4\alpha \sin^3\phi \cos^2\theta\, \mathrm{d}\theta\, \mathrm{d}\phi\, \mathrm{d}\alpha = 1, \end{aligned} \tag{5.8}$$

where the spherical coordinates $x_1 = \sin\alpha \sin\phi \cos\theta$, $x_2 = \sin\alpha \sin\phi \sin\theta$, $x_3 = \sin\alpha \cos\phi$, $x_4 = \cos\alpha$ were introduced. Thus we have verified equations (3.3) and (3.10) together with the linking number interpretation of λ.

To obtain the corresponding magnetic surfaces, we employ the Hopf map $\pi\colon S^3 \to S^2$ [29]. Parametrizing S^3 by $z_1 = \cos\frac{1}{2}\theta\, \mathrm{e}^{\mathrm{i}\Psi_1}$, $z_2 = \sin\frac{1}{2}\theta\, \mathrm{e}^{\mathrm{i}\Psi_2}$, we have

$$\begin{aligned}\pi(z_1, z_2) &= (z_1^* z_2 + z_2^* z_1, -\mathrm{i}z_1^* z_2 + \mathrm{i}z_2^* z_1, |z_1|^2 - |z_2|^2)\\ &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),\end{aligned} \tag{5.9}$$

where $\phi = \Psi_2 - \Psi_1$. If we set $(x_1, x_2, x_3, x_4) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta, 0)$ as the coordinates of the magnetic surface in S^3, we find that

$$\Delta x_i = -2x_i \qquad (i = 1, 2, 3, 4). \tag{5.10}$$

This verifies equation (5.5) and, of course, the surface is a patch on S^2.

In 1970, Lawson [30] showed that there are minimal embeddings into S^3 of surfaces of arbitrary genus. Since we have observed that the Berry phase may be associated with precisely these embeddings, it will be interesting to obtain new examples of the Berry phase that do not have the monopole structure noted above and which is so pervasive in the literature. The system in the next section is one such example.

6. Clifford torus

We describe in this section an example of the Berry phase that has the structure of the Clifford torus. This concerns the interaction between electrons and the nuclear displacements of an octahedral complex ML_6 (M is the metal ion and L the ligand). An octahedral Cu^{2+} complex has a d^9 configuration (single hole in a closed d shell). The five 3d orbitals are degenerate. When in the environment of an octahedron of negative charges, the d electrons split into a doublet and a triplet with the former lying higher. The hole goes into the doublet and, as a consequence, into an E_{g} representation of the O_{h} symmetry group [31, 32].

For strong ion–vibration coupling, the potential energy of the electron doublet paired with a doubly degenerate vibration has the form [31]

$$V = \tfrac{1}{2}\omega^2\big(R_\theta^2 + R_\varepsilon^2\big)1 + V_0(R_\theta\sigma_x + R_\varepsilon\sigma_y), \tag{6.1}$$

where R_θ and R_ε are real normal vibration coordinates, 1 is the unit 2×2 matrix and $\boldsymbol{\sigma}$ are the Pauli matrices. The full Schrödinger equation may be written symbolically as

$$[T_{\mathrm{nucl}}(R) + T_{\mathrm{el}}(r) + V(r, R)]\Psi = E\Psi, \tag{6.2}$$

r, R being electronic and nuclear coordinates. One now assumes that Ψ can be separated into nuclear and electronic components

$$\Psi(r, R) = \sum_n \Phi_n(r)\phi_n(r, R), \tag{6.3}$$

where n is a label for electronic eigenstates for fixed nuclear background. From the start we exclude the kinetic energy of the electron doublet so, for fixed R, we require that

$$V_0 R \begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\varphi} \\ \mathrm{e}^{\mathrm{i}\varphi} & 0 \end{pmatrix} \phi_n(r, R) = \varepsilon_n(R)\phi_n(r, R), \tag{6.4}$$

wherein the polar coordinates $R_\theta = R\cos\varphi$, $R\varepsilon = R\sin\varphi$ have been employed. Thus we have the solutions $\phi_{n\pm}^T = (\mathrm{e}^{-\mathrm{i}\varphi/2}, \pm\mathrm{e}^{\mathrm{i}\varphi/2})$ with eigenvalues $\pm V_0 R$. The full nuclear kinetic energy operator is $T_{\mathrm{nucl}} = -\frac{1}{2}\nabla_R^2$ (mass normalized to unity), so the Schrödinger equation,

$$\sum_n \left(T_{\mathrm{nucl}} + \frac{1}{2}\omega^2 R^2 + \varepsilon_n\right)\Phi_n(R)\phi_n(r, R) = E\sum_n \Phi_n(R)\phi_n(r, R), \tag{6.5}$$

may be simplified by integrating away the electron degrees of freedom. This is the way the Born–Oppenheimer approximation is obtained [2]. The result is

$$\left\{-\frac{1}{2}\sum_k [\delta_{mk}\nabla_R - \mathrm{i}A_{mk}(R)][\delta_{kn}\nabla_R - \mathrm{i}A_{kn}(R)] + \delta_{mn}\frac{1}{2}\omega^2 R^2 + \delta_{mn}\varepsilon_n(R)\right\} = E\Phi_n(R), \tag{6.6}$$

where the Berry gauge potential $A_{mn} = \mathrm{i}\langle\phi_m|\nabla_R\phi_n\rangle$ was introduced and has the simple form

$$\boldsymbol{A} = \frac{1}{R}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\hat{\phi}. \tag{6.7}$$

The nuclear system behaves like a charged particle in a magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A} = 0$. Although the magnetic field vanishes, we know, by the Aharonov–Bohm effect, that a phase will be picked up by the system and this is just the Berry phase. Ham [33] has argued that the order of the lowest vibronic levels of a Jahn–Teller system for an orbital doublet is specified by the requirement that the vibrational part of the wavefunction change sign under a 2π rotation in the vibration coordinates. Since the total wavefunction must be single valued in the space of vibration coordinates, this sign change, in turn, forces a compensating sign change in the electronic part of the wavefunction. This is precisely the Berry phase given above. Its structure is clearly not that of the monopole sort. Unlike the monopole case, its linking number is zero. We can understand now why $\boldsymbol{B}$ vanishes. Let us now give some topological details about this example.

The two-level system we are dealing with involves two-dimensional complex vectors. Hence the adiabatic transport of an eigenstate of the Hamiltonian defines a map from a submanifold of the nuclear R-space to the space of two-dimensional complex vectors of unit modulus. This is just the 3-sphere S^3. Since the magnetic field vanishes, curvature is zero and we have a flat submanifold in S^3, namely the Clifford torus T^2 given by the isometric immersion x: $\mathrm{R}^2 \to S^3 \subset \mathrm{R}^4$:

$$x(\theta,\varphi) = (\cos\theta, \sin\theta, \cos\varphi, \sin\varphi)/\sqrt{2} \tag{6.8}$$

with metric given by $\mathrm{d}s^2 = \frac{1}{2}(\mathrm{d}\theta^2 + \mathrm{d}\varphi^2)$ which is Euclidean and hence flat. The path followed by the E doublet is along a meridian $\theta =$ constant. It is known that the Clifford torus is minimal in S^3 [34].

7. Uniqueness of the gauge fields

We show in this section that gauge fields in S^3 that have the same Hopf invariant are equivalent up to a gauge transformation. A Riemannian metric will be understood to have been introduced. Thus, suppose that two gauge fields A and B exist such that

$$\int_{S^3} A\,\mathrm{d}A = \int_{S^3} B\,\mathrm{d}B = n. \tag{7.1}$$

It is easy to verify from this and the fact that S^3 is closed that

$$\int_{S^3} (A+B)\,\mathrm{d}(A-B) = 0. \tag{7.2}$$

We now show that for arbitrary fields ω and ϖ in S^3 satisfying the condition

$$\int \omega\,\mathrm{d}\varpi = 0, \tag{7.3}$$

one of them must vanish or be exact.

Any gauge field ω in a compact and closed manifold can be written as a sum of its harmonic part ω^0, its longitudinal part $\mathrm{d}\alpha$ and its transverse part $\delta\beta$. By virtue of equation (3.1b), the first two vanish so we have simply $\omega = \delta\beta$, i.e., ω is transverse. Now we expand ω in terms of a set of a normalized basis of transverse eigenforms φ_n of the Laplacian Δ : $\Delta\phi_n = \lambda_n^2\varphi_n, \delta\varphi_n = 0, (\varphi_n, \varphi_m) = \delta_{nm}$. The eigenvalues are non-vanishing. Thus

$$\omega = \sum_{n=1}^{\infty} c_n\varphi_n, \tag{7.4}$$

where c_n are expansion coefficients. We can also define another normalized basis of transverse eigenforms ϕ_n for ϖ:

$$\phi_n = -\frac{1}{\lambda_n}\,{}^*\mathrm{d}\varphi_n. \tag{7.5}$$

This is possible because the Hodge dual provides a natural isomorphism between the space of 1-forms and the corresponding space of 2-forms in compact 3-space. Observe also that $^*\mathrm{d}$ is linear and invertible. We can verify that $\Delta\phi_n = \lambda_n^2\phi_n,\ (\phi_n, \phi_m) = \delta_{nm}, \delta\phi_n = 0$. Hence we may write a parallel expansion for ϖ:

$$\varpi = \sum_{n=1}^{\infty} b_n\phi_n. \tag{7.6}$$

We now evaluate equation (7.3):

$$0 = \int_{S^3} \omega\,\mathrm{d}\varpi = \sum_{n,m} c_n b_m \int_{S^3} \varphi_n\,\mathrm{d}\phi_m = \sum_n c_n b_n \lambda_n, \tag{7.7}$$

where we used the various properties of the eigenforms defined above. Since the eigenvalues λ_n are non-vanishing and can be arranged in ascending order, this result holds if and only if all the expansion coefficients c_n or all the expansion coefficients b_n vanish. Therefore equation (7.2) holds if and only if

$$A = \pm B. \tag{7.8}$$

Because the Hopf invariant is gauge invariant we may relax this condition to

$$A = \pm B + \mathrm{d}f, \tag{7.9}$$

where f is a smooth 0-form. This shows that two gauge fields satisfying equation (7.1) are essentially gauge equivalent.

Two final remarks are in order. First, in section 5 we saw that a three-dimensional space (S^3 for instance) in which $A\,\mathrm{d}A = 0$ everywhere has zero magnetic field. Clearly, this field has zero linking number. We showed in section 6 that the Clifford torus also has zero linking number. We know now that these two fields (and any fields with zero linking number) are gauge equivalent. It follows that gauge fields with zero linking number in S^3 must necessarily have vanishing magnetic field. Second, the argument above was cast for three-dimensional space. It can be extended to any odd dimensional compact space. The only modification is in definition (7.5), which should read instead

$$\phi_n = (-)^{2p-1-p-1}\frac{1}{\lambda_n}\,{}^*\mathrm{d}\varphi_n \tag{7.10}$$

for p-eigenforms in $(2p-1)$-dimensional compact manifold.

8. Remark on stability

We observed in section 1 that most minimal surfaces are really unstable, that is, their second variation is negative. We will assume that the gauge field undergoes transport by the vector X while the surface area of the corresponding minimal surface undergoes normal variation. In section 3, we noted that the energy density $B \wedge {}^*B$ is invariant under X, so we need to consider only normal variations of the surface area $\int d\sigma$. It is known that $\delta \int d\sigma = -2 \int HV\, d\sigma$, where V is the normal variation of the surface $\boldsymbol{x}(u, v) \to \boldsymbol{x}(u, v) + tV(u, v)\hat{n}$, $t \in (-\varepsilon, \varepsilon)$. Because H vanishes for minimal surfaces, we see that the first variation of the surface area vanishes and only the second variation of the area is required (the energy density remaining invariant):

$$\delta^2 E_B = -2(B \wedge {}^*B\delta H). \tag{8.1}$$

For compact and closed 2-manifolds in S^3 it can be shown that [35]

$$\delta H = \tfrac{1}{2}(\Delta V + 2KV), \tag{8.2}$$

where $K = 1$ is the constant sectional curvature of S^3 and Δ the Laplacian on the surface. Limiting ourselves to normal variations proportional to the coordinates of the surface, by virtue of equation (5.5), we find $\delta H = 0$ and so for gauge fields in S^3

$$\delta^2 E_B = 0. \tag{8.3}$$

For the case of the helicoidal field in R^3, which is non-compact, the corresponding normal variation of H is [27]

$$\delta H = \tfrac{1}{2}(\Delta V + 4H^2 V - 2KV), \tag{8.4}$$

where $H = 0$ for a helicoid and K (< 0) is its Gaussian curvature. Since minimal surfaces in R^3 satisfy $\Delta x = 0$, we find then that

$$\delta^2 E_B < 0. \tag{8.5}$$

Thus, we have established the instability of the helicoid, but gauge fields in S^3 appear to require further study. It would be necessary to consider more general surface and field variations.

The instability of a helicoidal soap film can be observed as follows: one forms a contour of two identical spirals winding on opposite sides of their common axis and closed by two horizontal wires at their ends. This is used as a frame for a soap bubble. If the spirals' pitch is great, the soap film stretched across this frame is a helicoid. As the pitch is decreased the spirals contract and a point is reached when the soap film ceases to be a helicoid, becoming instead a film resembling a strip wound helically about a cylinder [24]. The helicoidal film has undergone a change of topological type.

Turning now to the Berry phase, in their experiment on a beam of spin-aligned neutrons passing through a helical magnetic field of variable pitch, Bitter and Dubbers [36] reported good agreement with theory for large pitch but less than ideal agreement for small pitch. They attributed the latter to the failure of the adiabatic approximation. We surmise, however, that this is really a manifestation of the instability of the helicoid and a change of topology.

9. Conclusions

We have shown that the symmetry-inspired equations (3.1) may be used to understand features associated with the Berry phase. We also saw that the topology of the parameter manifold may be expanded to include non-compact spaces, that the magnetic surface corresponding to the Berry phase is a minimal surface and that the topological structure of these surfaces is nontrivial and may be classified by the Hopf invariant. The now classic Berry phase is really a circle-bundle of the 2-sphere forming S^3; we have noted that there may exist an infinite variety of minimal surfaces embedded in S^3 and that, correspondingly, there should be an infinite variety of Berry phases associated with these surfaces. We also showed that gauge fields may be classified by their Hopf invariants. The issue of instability appears a fruitful area for future study. This may also be pursued in parallel with a study of non-Abelian Berry phases [37].

References

[1] Berry M V 1984 *Proc. R. Soc. Lond.* A **392** 45
[2] Shapere A and Wilczek F 1989 *Geometric Phases in Physics* (Singapore: World Scientific)
Bohm A, Mostafazadeh A, Koizumi H and Zwanziger J 2003 *The Geometric Phase in Quantum Systems* (Heidelberg: Springer)
[3] Marsden J E, Montgomery R and Ratiu R 1990 *Mem. Am. Math. Soc.* **88** 436
[4] Ferraro V C A and Plumpton C 1961 *An Introduction to Magneto-Fluid Mechanics* (Oxford: Oxford University Press)
[5] Aly J J 1984 *Astrophys. J.* **284** 349
[6] O'Neil B 1983 *Semi-Riemannian Geometry* (London: Academic) chapter 4
[7] Finkelstein D and Rubinstein J 1968 *J. Math. Phys.* **9** 1762
[8] Bargmann V 1964 *J. Math. Phys.* **5** 862
[9] Simon R and Mukunda N 1993 *Phys. Rev. Lett.* **70** 880
[10] Aharonov Y and Anandan J 1987 *Phys. Rev. Lett.* **58** 1593
[11] Papadopoulous G 1992 *J. Phys. A: Math. Gen.* **25** 2071
[12] Harnad J, Shnider S and Vinet L 1979 *J. Math. Phys.* **20** 931
[13] Nowakowski J and Trautman A 1978 *J. Math. Phys.* **19** 1100
[14] Eguchi T, Gilkey P and Hanson A 1980 *Phys. Rep.* **66** 213
[15] Berger M A and Field G B 1984 *J. Fluid Mech.* **147** 133
[16] Samuel J and Bhaduri R 1988 *Phys. Rev. Lett.* **60** 2339
[17] Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry* vol 1 (New York: Interscience) p 34
[18] Wilczek F and Zee A 1983 *Phys. Rev. Lett.* **51** 2250
[19] Wu Y-S and Zee A 1984 *Phys. Lett.* B **147** 325
[20] Bowick M, Karabali D and Wijewardhana L 1986 *Nucl. Phys.* B **271** 417
[21] Arnold V I 1986 *Sel. Math. Sov.* **5** 327
[22] Moffat K 1985 *J. Fluid Mech.* **159** 339
[23] Gray A 1993 *Modern Differential Geometry of Curves and Surfaces* (Boca Raton, FL: CRC Press) p 345
[24] Fomenko A T and Tuzhilin A A 1991 *Elements of the Geometry and Topology of Minimal Surfaces in Three-Dimensional Space* (*TMN* vol 93) (Providence, RI: American Mathematical Society) p 39
[25] Segert J 1987 *Phys. Rev.* A **36** 10
[26] Lima E L 1965 *Ann. Math.* **81** 70
[27] Willmore T J 1982 *Total Curvature in Riemannian Geometry* (Chichester: Ellis Horwood) chapter 5
[28] Trautman A 1977 *Int. J. Theor. Phys.* **16** 561
[29] Gockeler M and Schucker T 1987 *Differential Geometry, Gauge Theories and Gravity* (Cambridge: Cambridge University Press) chapter 10
[30] Lawson H B 1970 *Ann. Math.* **92** 335
[31] Bersuker I B and Polinger V Z 1989 *Vibronic Interactions in Molecules and Crystals* (Berlin: Springer) pp 33 and 120
[32] Inui T, Tanabe Y and Onodera Y 1990 *Group Theory and its Applications in Physics* (Berlin: Springer) section 10.4
[33] Ham F 1987 *Phys. Rev. Lett.* **58** 725
[34] Nitsche J C C 1989 *Lectures on Minimal Surfaces* vol 1 (Cambridge: Cambridge University Press) p 80
[35] Weiner J L 1978 *Indiana Univ. Math. J.* **27** 19
[36] Bitter T and Dubbers D 1987 *Phys. Rev. Lett.* **59** 251
[37] Marzlin K-P, Bartlett S and Sanders B 2003 *Phys. Rev.* A **67** 022316